Solutions for f(R) gravity coupled with electromagnetic field

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In the presence of external, linear / nonlinear electromagnetic fields we integrate $f(R) \sim R + 2\alpha\sqrt{R + const.}$ gravity equations. In contrast to their Einsteinian cousins the obtained black holes are non-asymptotically flat with a deficit angle. In proper limits we obtain from our general solution the global monopole solution in $f(R)$ gravity. The scale symmetry breaking term adopted as the nonlinear electromagnetic source adjusts the sign of the mass of the resulting black hole to be physical.

I. INTRODUCTION

$f(R)$ gravity is a modified version of standard Einstein's gravity which incorporates an arbitrary function of the Ricci scalar (R) instead of the linear one [1] (for a recent review). Depending only on the Ricci scalar may sound simpler at the initial but the pertinent nonlinearity makes nothing simpler than the Einstein's gravity with sources. There are both advantages and disadvantages in adopting such a model. It contains for instance, its own source known as the curvature source in the absence of an external matter source. The identification of physical sources, however, within the nonlinear structure through its equations is not an easy task at all. For the same reason almost all known solutions, except very few, result in nonanalytical (i.e. numerical) expressions for the function $f(R)$. Starting from a known function of $f(R)$ a priori is an alternative approach which hosts its own shortcoming from the outset. Keeping a set of free parameters to be fixed by observational data can be employed in favour of $f(R)$ gravity to explain a number of cosmological phenomena. First of all, to be on the safe side along with the successes of general relativity most researchers prefer an ansatz of the form $f(R) = R + \alpha g(R)$, so that with $\alpha \to 0$ one recovers the Einstein limit. The struggle now is for the new function $g(R)$ whose equations are not easier than those satisfied by $f(R)$ itself. Without seeking resort to this latter (and easier) route we have shown recently that $f(R) = \sqrt{R}$ gravity admits exact solution in $6-$dimensional spacetime with the external Yang-Mills field [2]. Without demanding an analytical representation for $f(R)$, as a matter of fact, exact solutions are available in all dimensions with the Yang-Mills source. Similar results may be investigated with other sources such as the Maxwell fields. This will be our strategy in the present Letter.

We assume $f(R) = \xi(R + R_1) + 2\alpha\sqrt{R + R_0}$, in which ξ, α, R_0 and R_1 are constants, a priori to secure the Einstein limit by setting the constants $R_0 = R_1 = \alpha = 0$ and $\xi = 1$. This extends a previous study without sources [3] to the case with sources. Why the square root term in the Lagrangian?. It will be shown that for $R_0 = R_1 = 0$ and without external sources such a choice of square root Lagrangian gives the curvature energy-momentum tensor components as $T_t^t = T_r^r$, $T_\theta^\theta = T_\varphi^\varphi = 0$, which signify a global monopole [4]. A global monopole which arises from spontaneous breaking of gauge symmetry is the minimal structure that yields non-zero curvature even with zero mass. We test the analogous concept in $f(R)$ gravity to obtain similar structures. Unlike the case of [2] our concern here will be restricted to the $4-$dimensional spacetime. As source, we take electromagnetic fields, both from the linear (Maxwell) and the nonlinear theories. For the linear Maxwell source we obtain a black hole solution with electric charge (Q) and magnetic charge (P) reminiscent of the Reissner-Nordstrom (RN) solution with different asymptotic behaviors. That is, our spacetime is non-asymptotically flat with a deficit angle. For the nonlinear, pure electric source we choose the standard Maxwell invariant superposed with the square root invariant, i.e. the Lagrangian is given by $\mathcal{L}(F) \sim F + 2\beta\sqrt{-F}$, where $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant and β is a coupling constant. This particular choice has the feature that it breaks the scale invariance [5] , gives a linear electric potential which plays role in quark confinement [6, 7]. We find out that the scale breaking parameter β modifies the mass of the black hole. For this reason Lagrangians supplemented by a square-root Maxwell Lagrangian may find rooms of applications in black hole physics.

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II. $f(R)$ GRAVITY COUPLED WITH MAXWELL FIELD

The action for $f(R)$ gravity coupled with Maxwell field in 4-dimensions is given by

$$S=\int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa}-\frac{1}{4\pi}F\right] \tag{1}$$

in which $f(R)$ is a real function of Ricci scalar R and $F=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant. (We choose $\kappa=8\pi$ and $G=1$). The Maxwell two-form is chosen to be

$$\mathbf{F}=\frac{Q}{r^2}dt\wedge dr+P\sin\theta d\theta\wedge d\phi \tag{2}$$

in which Q and P are the electric and magnetic charges, respectively. Our static spherically symmetric metric ansatz is

$$ds^2=-A(r)\,dt^2+\frac{dr^2}{A(r)}+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{3}$$

where $A(r)$ stands for the only metric function to be found. The Maxwell equations (i.e. $dF=0=d^{*}F$) are satisfied and the field equations are given by

$$f_R R_\mu^\nu+\left(\Box f_R-\frac{1}{2}f\right)\delta_\mu^\nu-\nabla^\nu\nabla_\mu f_R=\kappa T_\mu^\nu \tag{4}$$

in which

$$\begin{aligned} f_R &= \frac{df(R)}{dR}, && (5)\\ \Box f_R &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\right)f_R, && (6)\\ \nabla^\nu\nabla_\mu f_R &= g^{\alpha\nu}\left[(f_R)_{,\mu,\alpha}-\Gamma_{\mu\alpha}^m(f_R)_{,m}\right], && (7)\end{aligned}$$

while the energy momentum tensor is

$$4\pi T_\mu^\nu=-F\delta_\mu^\nu+F_{\mu\lambda}F^{\nu\lambda}. \tag{8}$$

Furthermore, the trace of the field equation (4) reads

$$f_R R+(d-1)\Box f_R-\frac{d}{2}f=\kappa T \tag{9}$$

with $T=T_\mu^\mu$. The non-zero energy momentum tensor components are

$$T_\mu^\nu=\frac{P^2+Q^2}{8\pi r^4}diag\left[-1,-1,1,1\right] \tag{10}$$

with zero trace and consequently

$$f=\frac{1}{2}f_R R+3\Box f_R. \tag{11}$$

One finds

$$\begin{aligned} R &= -\frac{r^2A''+4rA'+2(A-1)}{r^2}, && (12)\\ R_t^t &= R_r^r=-\frac{1}{2}\frac{rA''+2A'}{r}, && (13)\\ R_\theta^\theta &= R_\phi^\phi=-\frac{rA'+A-1}{r^2}. && (14)\end{aligned}$$

in which a prime denotes derivative with respect to r. Overall, the field equations read now

$$f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^t\nabla_t f_R \;=\; \kappa T_0^0, \tag{15}$$

$$f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^r\nabla_r f_R \;=\; \kappa T_1^1, \tag{16}$$

$$f_R\left(-\frac{rA'+(A-1)}{r^2}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^\theta\nabla_\theta f_R \;=\; \kappa T_2^2. \tag{17}$$

Herein

$$\Box f_R = A'f_R'+Af_R''+\frac{2}{r}Af_R',\ \nabla^t\nabla_t f_R=\frac{1}{2}A'f_R',\ \nabla^r\nabla_r f_R=Af_R''+\frac{1}{2}A'f_R',\ \nabla^\phi\nabla_\phi f_R=\nabla^\theta\nabla_\theta f_R=\frac{A}{r}f_R' \tag{18}$$

and for the details we refer to [2]. The tt and rr components of the field equations imply

$$\nabla^r\nabla_r f_R=\nabla^t\nabla_t f_R \tag{19}$$

or equivalently

$$f_R''=0. \tag{20}$$

This leads to the solution

$$f_R=\xi+\eta r \tag{21}$$

where ξ and η are two positive constants [8]. The other field equations become

$$f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\frac{1}{2}\eta A'+\frac{2}{r}A\eta-\frac{1}{2}f \;=\; \kappa T_0^0, \tag{22}$$

$$f_R\left(-\frac{rA'+(A-1)}{r^2}\right)+A'\eta+\frac{1}{r}A\eta-\frac{1}{2}f \;=\; \kappa T_2^2. \tag{23}$$

Now, we make the choice

$$f(R)=\xi\left(R+\frac{1}{2}R_0\right)+2\alpha\sqrt{R+R_0} \tag{24}$$

which leads to

$$R=\frac{\alpha^2}{\eta^2 r^2}-R_0 \tag{25}$$

where α, R_0 and ξ (from (21)) are constants. As a result one obtains for $f(r)$

$$f=\frac{\xi\alpha^2}{\eta^2 r^2}+\frac{2\alpha^2}{\eta r}-\frac{1}{2}\xi R_0 \tag{26}$$

and from (12) we have

$$-\frac{r^2A''+4rA'+2(A-1)}{r^2}=\frac{\alpha^2}{\eta^2 r^2}-R_0. \tag{27}$$

This equation admits a solution for the metric function given by

$$A(r)=1-\frac{\alpha^2}{2\eta^2}+\frac{C_1}{r}+\frac{C_2}{r^2}+\frac{1}{12}R_0 r^2. \tag{28}$$

Herein the two integration constants C_1 and C_2 are identified through the other field equations (22) and (23) as

$$C_1=\frac{\xi}{3\eta}\ \text{and}\ C_2=\frac{\left(Q^2+P^2\right)}{\xi}, \tag{29}$$

while for the free parameters we have $\alpha=\eta>0$. Finally the metric function becomes

$$A\left(r\right)=\frac{1}{2}-\frac{m}{r}+\frac{q^{2}}{r^{2}}-\frac{\Lambda_{eff}}{3}r^{2} \tag{30}$$

where $m=-\frac{\xi}{3\eta}<0$, $\Lambda_{eff}=\frac{-R_0}{4}$ and $q^2=\frac{\left(Q^2+P^2\right)}{\xi}$. The choice of the free parameters in terms of each other prevents us from obtaining the general relativity limit, namely the Reissner-Nordström (RN)-de Sitter (dS) solution. It is observed that the parameter ξ acts as a scale factor for mass and charge and for the case $\xi=1$ and $Q=P=0$ the solution reduces to the known solution given by [3, 9]. The properties of this solution can be summarized as follow: The mass term has the opposite sign to that of Schwarzschild and the solution is not asymptotically flat, giving rise to a deficit angle. The latter property is reminiscent of a global monopole term with a fixed charge. To see the case of a global monopole we set $R_0=0=q^2$ (i.e. zero external charges and zero cosmological constant) and find the energy-momentum components. This reveals that the non-zero components are $T_t^t=T_r^r=-\frac{1}{2r^2}$, which identifies a global monopole [4]. The solution (30) can therefore be interpreted as an Einstein-Maxwell plus a global monopole solution in $f(R)$ gravity. The area of a sphere of radius r (for $q^2=R_0=0$) is not $4\pi r^2$ but $2\pi r^2$. Further, it can be shown easily that the surface $\theta=\frac{\pi}{2}$ has the geometry of a cone with a deficit angle $\Delta=\frac{\pi}{2}$ [4]a. It can also be anticipated that a global monopole modifies perihelion of circular orbits, light bending and other physical properties. Although in the linear Maxwell theory the sign of mass is opposite, in the next section we shall show that this can be overcome by going to the nonlinear electrodynamics with a square root Lagrangian. Another aspect of the solution is that since $f_R>0$ we have no ghost states.

III. $f(R)$ GRAVITY COUPLED WITH NONLINEAR ELECTROMAGNETISM

A. Solution within nonlinear electrodynamics

In this section we use an extended model for the Maxwell Lagrangian given in the action

$$S=\int d^4x\sqrt{-g}\left[\frac{f\left(R\right)}{2\kappa}+\mathcal{L}\left(F\right)\right] \tag{31}$$

where $f\left(R\right)=\xi\left(R+R_1\right)+2\alpha\sqrt{R+R_0}$, in which R_1 and R_0 are constants to be found while

$$\mathcal{L}\left(F\right)=-\frac{1}{4\pi}\left(F+2\beta\sqrt{-F}\right). \tag{32}$$

Here β is a free parameter such that $\lim_{\beta\to0}\mathcal{L}\left(F\right)=-\frac{1}{4\pi}F$, which is the linear Maxwell Lagrangian. The main reason for adding this term is to break the scale invariance and create a mass term [4]. The normal Maxwell action is known to be invariant under the scale transformation, $x\to\lambda x$, $A_\mu\to\frac{1}{\lambda}A_\mu$, ($\lambda$ =const.), while $\sqrt{-F}$ violates this rule. We shall show how a similar term modifies the mass term in $f(R)$ gravity. Our choice of the Maxwell 2-form is written as

$$\mathbf{F}=E\left(r\right)dt\wedge dr \tag{33}$$

and the spherical line element as (3). The nonlinear Maxwell equation reads

$$d\left({}^{\star}\mathbf{F}\frac{\partial\mathcal{L}}{\partial F}\right)=0 \tag{34}$$

which yields the solution

$$E\left(r\right)=\sqrt{2}\beta+\frac{Q}{r^2} \tag{35}$$

with a confining electric potential as $V\left(r\right)=-\sqrt{2}\beta r+\frac{Q}{r}$. This is known as the "Cornell potential" for quark confinement in quantum chromodynamics (QCD) [6, 7]. The Einstein equations implies the same equations as (4-7) and the energy momentum tensor

$$\begin{aligned}T_\mu^\nu &= \mathcal{L}\left(F\right)\delta_\mu^\nu-F_{\mu\lambda}F^{\nu\lambda}\frac{\partial\mathcal{L}}{\partial F}= \\ &\frac{F}{4\pi}\mathrm{diag}\left[1,1,\frac{2\beta}{\sqrt{-F}}-1,\frac{2\beta}{\sqrt{-F}}-1\right],\end{aligned} \tag{36}$$

with the additional condition that the trace $T_{\mu}^{\mu}=T\neq 0$, here. Upon substitution into the field equations one gets

$$R_1 = \frac{4\beta^2}{\xi}+\frac{1}{2}R_0. \tag{37}$$

$$\alpha = \eta \tag{38}$$

and a black hole solution results with the metric function

$$A(r)=\frac{1}{2}-\frac{4\sqrt{2}\beta Q-\xi}{3\eta r}+\frac{Q^2}{\xi r^2}+\frac{R_0}{12}r^2. \tag{39}$$

This is equivalent to the solution given in (30) with the same Λ_{eff} but with the new $m=\frac{4\sqrt{2}\beta Q-\xi}{3\eta}$ and $q=\frac{Q^2}{\xi}$. This is how the scale breaking term in the Lagrangian modifies the mass.

For the sake of completeness we comment here that, choosing a magnetic ansatz for the field two-form as

$$\mathbf{F}=P\sin\theta d\theta\wedge d\varphi \tag{40}$$

together with a nonlinear Maxwell Lagrangian

$$\mathcal{L}(F)=-\frac{1}{4\pi}\left(F+2\beta\sqrt{F}\right) \tag{41}$$

and

$$R_1=\frac{1}{2}R_0 \tag{42}$$

admits the magnetic version of the solution as

$$A(r)=\frac{1}{2}-\frac{4\sqrt{2}\beta P-\xi}{3\eta r}+\frac{P^2}{\xi r^2}+\frac{R_0}{12}r^2. \tag{43}$$

The magnetic solution, however, is not as interesting as the electric one.

B. Thermodynamical aspects

The solution we found in the previous section is feasible as far as a physical solution is concerned. Here we set our parameters, including the condition ξ and η positive, to get $4\sqrt{2}\beta Q-\xi>0$ such that the solution admits a black hole solution with positive mass as

$$A(r)=\frac{1}{2}-\frac{m}{r}+\frac{q^2}{r^2}+\frac{R_0}{12}r^2. \tag{44}$$

Now we wish to discuss some of the thermodynamical properties by using the Misner-Sharp [2, 10] energy to show that the first law of thermodynamics is satisfied. To do so first we set $R_0=0$ and introduce the possible event horizon as $r=r_h$ such that $A(r_h)=0$. This yields

$$r_{\pm} = m\pm\sqrt{m^2-2q^2} \tag{45}$$
$$(r_h=r_+)$$

in which

$$A(r)=\frac{(r-r_-)(r-r_+)}{2r^2} \tag{46}$$

and the constraint $m\geq m_{cri}$ is imposed with $m_{crit}=\sqrt{2}q$. If one sets $Q>0$, this condition is satisfied if $Q>\frac{\xi}{\sqrt{2}\left(4\beta+\frac{3}{\sqrt{\xi}\eta}\right)}$ (providing $4\beta+\frac{3}{\sqrt{\xi}\eta}\neq 0$). The choice $m=m_{crit}$ leads to the extremal black hole. The Hawking temperature is defined as

$$T_H=\frac{A'(r_+)}{4\pi}=\frac{r_+^2-2q^2}{8\pi r_+^3} \tag{47}$$

and the entropy [11]

$$S = \frac{\mathcal{A}_+}{4G} \left. f_R \right|_{r=r_+} \tag{48}$$

with $\mathcal{A}_+ = 4\pi r_+^2$, the surface area of the black hole at the horizon. The heat capacity of the black hole also is given by

$$C_q = T \left(\frac{dS}{dT} \right)_q = -\frac{2}{3} \frac{r_+^2 \pi \left(2q^2 - r_+^2\right) \left(12q^4 + 4q^2 r_+^2 + r_+^4\right)}{\left(2q^2 + r_+^2\right)^2 \left(6q^2 - r_+^2\right)}. \tag{49}$$

which takes both $(+)$ and $(-)$ values. Both the vanishing / diverging C_q values indicate special points at which the system attains thermodynamical phrase changes.

The first law of thermodynamics can be written as

$$TdS - dE = PdV \tag{50}$$

in which

$$dE = \frac{1}{2\kappa} \left[\frac{2}{r_h^2} f_R + (f - R f_R) \right] \mathcal{A}_+ dr_+ \tag{51}$$

with E the Misner-Sharp energy and $T = \frac{A'}{4\pi}$ the Hawking temperature. Further, $S = \frac{\mathcal{A}_+}{4} f_R$ stands for the black hole entropy, $p = T_r^r = T_0^0$ is the radial pressure of matter fields at the horizon and finally the change of volume of the black hole at the horizon is given by $dV = \mathcal{A}_+ dr_+$. One can easily show that the first law in thermodynamics in the form introduced above is satisfied.

IV. CONCLUSION

Exact solutions for nowadays popular, modified gravity model known as $f(R)$ gravity with external sources (i.e. $T_{\mu\nu}^{matter} \neq 0$) are rare in the literature. We attempt to fill this vacuum partially by considering external electromagnetic fields (both linear and nonlinear) in $f(R)$ gravity with the ansatz $f(R) = \xi (R + R_1) + 2\alpha\sqrt{R + R_0}$. In this choice R_0 is a constant related to the cosmological constant, the constant R_1 is related to R_0 while α is the coupling constant for the correction term. This covers both the cases of linear Maxwell and a special case of power-law nonlinear electromagnetism. The non-asymptotically flat black hole solution obtained for the Maxwell source is naturally different and has no limit of the RN black hole solution. In the limit of $Q = P = \Lambda_{eff} = 0$ we obtain the metric for a global monopole in $f(R)$ gravity. Our solution can appropriately be interpreted as a global monopole solution in the presence of the electromagnetic fields. The thermodynamical properties of our black hole solution is analyzed by making use of the Misner-Sharp formalism and shown to obey the first law. As the nonlinear electromagnetic Lagrangian we choose the normal Maxwell, supplemented with the square root Maxwell invariant which amounts to a linear electric field. This latter form is known to break the scale invariance yielding a linear potential which is believed to play role in quark confinement problem. Within $f(R)$ gravity the presence of scale breaking term modifies the mass of the resulting black hole. The advantage of employing square-root Maxwell Lagrangian as a nonlinear correction can be stated as follows: Beside confinement in the linear Maxwell case we have in $f(R)$ gravity an opposite mass term while with the coupling of the aquare-root Maxwell Lagrangian we can rectify the sign of this term.

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